Sequences and series Mixed exercise 3

1 a Let a = first term and r = common ratio.

3rd term = 27 ⇒ $ar^2 = 27$ (**1**)

6th term = 8 ⇒ $ar^5 = 8$ (**2**)

Equation (**2**) ÷ Equation (**1**):

$$\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{8}{27} \quad \left(\frac{r^5}{r^2} = r^{5-2}\right)$$

$$r^3 = \frac{8}{27}$$

$$r = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

The common ratio is $\frac{2}{3}$.

b Substitute $r = \frac{2}{3}$ back into Equation (**1**):

$$a \times \left(\frac{2}{3}\right)^2 = 27$$

$$a \times \frac{4}{9} = 27$$

$$a = \frac{27 \times 9}{4}$$

$$a = 60.75$$

The first term is 60.75.

c Sum to infinity $= \frac{a}{1-r}$

$$\Rightarrow S_\infty = \frac{60.75}{1-\frac{2}{3}} = \frac{60.75}{\frac{1}{3}} = 182.25$$

Sum to infinity is 182.25.

d Sum to ten terms $\frac{a\left(1-r^{10}\right)}{1-r}$

So

$$S_{10} = \frac{60.75\left(1-\left(\frac{2}{3}\right)^{10}\right)}{\left(1-\frac{2}{3}\right)} = \frac{60.75\left(1-\left(\frac{2}{3}\right)^{10}\right)}{\frac{1}{3}}$$

$$= 179.0895\ldots$$

Difference between S_{10} and S_∞ = 182.25 − 179.0895 = 3.16 (3 s.f.)

2 a 2nd term is 80 ⇒ $ar^{2-1} = 80$

$ar = 80$ (**1**)

5th term is 5.12 ⇒ $ar^{5-1} = 5.12$

$ar^4 = 5.12$ (**2**)

Equation (**2**) ÷ Equation (**1**):

$$\frac{\cancel{a}r^4}{\cancel{a}r} = \frac{5.12}{80}$$

$$r^3 = 0.064 \left(\sqrt[3]{\ }\right)$$

$$r = 0.4$$

Hence common ratio = 0.4.

b Substitute $r = 0.4$ into Equation (**1**):

$$a \times 0.4 = 80\ (\div 0.4)$$

$$a = 200$$

The first term in the series is 200.

c $$S_\infty = \frac{a}{1-r} = \frac{200}{1-0.4} = \frac{200}{0.6} = 333\frac{1}{3}$$

2 d Sum to n terms $=\dfrac{a(1-r^n)}{1-r}$

So $S_{14}=\dfrac{200(1-0.4^{14})}{(1-0.4)}=333.3324385$

Required difference

$$S_{14}-S_\infty=333.3324385-333\frac{1}{3}$$
$$=0.0008947=8.95\times10^{-4}\,(3\text{ s.f.})$$

3 a $u_n=95\left(\dfrac{4}{5}\right)^n$

Replace n with 1 $\Rightarrow$ $u_1=95\left(\dfrac{4}{5}\right)^1=76$

Replace n with 2 $\Rightarrow$ $u_2=95\left(\dfrac{4}{5}\right)^2=60.8$

b Replace n with 21 $\Rightarrow$

$u_{21}=95\left(\dfrac{4}{5}\right)^{21}=0.876\ \ (3\text{ s.f.})$

c $$\sum_{n=1}^{15}u_n=\underbrace{76+60.8+\ldots+95\left(\frac{4}{5}\right)^{15}}_{15\text{ terms}}$$

A geometric series with a = 76, $r=\dfrac{4}{5}$.

Use $S_n=\dfrac{a(1-r^n)}{1-r}$

$$\sum_{n=1}^{15}u_n=\frac{76\left(1-\left(\frac{4}{5}\right)^{15}\right)}{1-\frac{4}{5}}=\frac{76\left(1-\left(\frac{4}{5}\right)^{15}\right)}{\frac{1}{5}}$$

$\left(\div\dfrac{1}{5}\text{ is equivalent to}\times5\right)$

$$\sum_{n=1}^{15}u_n=76\times5\times\left(1-\left(\frac{4}{5}\right)^{15}\right)$$
$$=366.63=367\ (\text{to }3\text{ s.f.})$$

d $S_\infty=\dfrac{a}{1-r}=\dfrac{76}{1-\frac{4}{5}}=\dfrac{76}{\frac{1}{5}}=76\times5=380$

Sum to infinity is 380.

4 a $u_n=3\left(\dfrac{2}{3}\right)^n-1$

Replace n with 1 $\Rightarrow$

$u_1=3\times\left(\dfrac{2}{3}\right)^1-1=2-1=1$

Replace n with 2 $\Rightarrow$

$u_2=3\times\left(\dfrac{2}{3}\right)^2-1=3\times\dfrac{4}{9}-1=\dfrac{1}{3}$

Replace n with 3 $\Rightarrow$

$u_3=3\times\left(\dfrac{2}{3}\right)^3-1=3\times\dfrac{8}{27}-1=-\dfrac{1}{9}$

4 b

$$\sum_{n=1}^{15} u_n = \left(3\times\left(\frac{2}{3}\right)-1\right)+\left(3\times\left(\frac{2}{3}\right)^2-1\right)$$

$$+\left(3\times\left(\frac{2}{3}\right)^3-1\right)+\ldots+\left(3\times\left(\frac{2}{3}\right)^{15}-1\right)$$

$$=\underbrace{3\times\left(\frac{2}{3}\right)+3\times\left(\frac{2}{3}\right)^2+3\times\left(\frac{2}{3}\right)^3+\ldots+3\times\left(\frac{2}{3}\right)^{15}}_{\text{a geometric series with 15 terms}}$$

$$\underbrace{-1-1-1-\ldots-1}_{\text{15 times}}$$

where $a = 3\times\frac{2}{3} = 2$ and $r = \frac{2}{3}$

Use $S_n = \frac{a(1-r^n)}{1-r}$

$$\sum_{n=1}^{15} u_n = \frac{2\left(1-\left(\frac{2}{3}\right)^{15}\right)}{1-\frac{2}{3}} - 15 = 5.986\ldots - 15$$

$$= -9.0137\ldots = -9.014 \text{ (4 s.f.)}$$

c

$$u_{n+1} = 3\times\left(\frac{2}{3}\right)^{n+1}-1$$

$$= 3\times\frac{2}{3}\times\left(\frac{2}{3}\right)^{n}-1$$

$$= 2\left(\frac{2}{3}\right)^{n}-1 = \frac{2u_n-1}{3}$$

5 a Let a = first term and r = the common ratio of the series.

We are given

3rd term = 6.4 ⇒ $ar^2 = 6.4$ **(1)**

4th term = 5.12 ⇒ $ar^3 = 5.12$ **(2)**

Equation **(2)** ÷ Equation **(1)**:

$$\frac{\cancel{a}r^3}{\cancel{a}r^2} = \frac{5.12}{6.4}$$

$$r = 0.8$$

The common ratio is 0.8.

b Substitute $r = 0.8$ into Equation **(1)**:

$$a\times 0.8^2 = 6.4$$

$$a = \frac{6.4}{0.8^2}$$

$$a = 10$$

The first term is 10.

c Use $S_\infty = \frac{a}{1-r}$ with $a = 10$ and $r = 0.8$.

$$S_\infty = \frac{10}{1-0.8} = \frac{10}{0.2} = 50$$

Sum to infinity is 50.

d

$$S_{25} = \frac{a(1-r^{25})}{1-r} = \frac{10(1-0.8^{25})}{1-0.8}$$

$$= 49.8111\ldots$$

$S_\infty - S_{25} = 50 - 49.8111\ldots$

$= 0.189$ (3 s.f.)

6 a $u_5 = 20\,000\times 0.85^5 = £8874.11$

b

$$20\,000\times 0.85^n < 4000$$

$$0.85^n < 0.2$$

$$n > \frac{\log 0.2}{\log 0.85}$$

$$n > 9.9$$

So the value will be less than £4000 after 9.9 years.

7 a $\frac{p(2q+2)}{p(3q+1)} = \frac{p(2q-1)}{p(2q+2)}$

$(2q+2)^2 = (2q-1)(3q+1)$

$4q^2 + 8q + 4 = 6q^2 - q - 1$

$2q^2 - 9q - 5 = 0$

$(q-5)(2q+1) = 0$

$q = 5$ or $q = -\frac{1}{2}$

b $q = 5$, $S_\infty = 896$, $a = 16p$, $r = 0.75$

$\frac{16p}{1-0.75} = 896$

$p = 14$

$a = 224$

$S_{12} = \frac{224(1-0.75^{12})}{1-0.75}$

$= 867.62$

8 a $S = a + (a + d) + (a + 2d) + \ldots + (a + (n - 2)d) + (a+(n - 1)d)$

Turning series around:

$S = (a + (n - 1)d) + (a + (n - 2)d) + \ldots (a + d) + a$

Adding the two sums:

$2S = (2a + (n - 1)d) + (2a + (n - 1)d) + \ldots (2a + (n - 1)d) + (2a + (n - 1)d)$

There are n lots of $(2a + (n - 1)d)$:

$2S = n \times (2a + (n - 1)d)$

$(\div 2)$: $S = \frac{n}{2}\left(2a + (n-1)d\right)$

b The first 100 natural numbers are 1, 2, 3, . . . 100.

We need to find
$S = 1 + 2 + 3 + \ldots 99 + 100$.

This series is arithmetic with $a = 1$, $d = 1$, $n = 100$.

Using $S = \frac{n}{2}\left(2a + (n-1)d\right)$ with $a = 1$, $d = 1$ and $n = 100$ gives

$S = \frac{100}{2}\left(2\times 1 + (100-1)\times 1\right)$

$= \frac{100}{2}\left(2 + 99\times 1\right)$

$= 50 \times 101 = 5050$

9 $\sum_{r=1}^{n}(4r-3) = (4\times 1 - 3) + (4\times 2 - 3) + (4\times 3 - 3) + \ldots + (4\times n - 3)$

$= \underbrace{1 + 5 + 9 + \ldots + (4n-3)}$

Arithmetic series with $a = 1$, $d = 4$.

Using $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$ with $a = 1$, $d = 4$ gives

$S_n = \frac{n}{2}\left(2\times 1 + (n-1)\times 4\right) = \frac{n}{2}\left(2 + 4n - 4\right)$

$= \frac{n}{2}(4n-2)$

$= n(2n-1)$

Solve $S_n = 2000$:

$n(2n - 1) = 2000$

$2n^2 - n = 2000$

$2n^2 - n - 2000 = 0$

$n = \frac{1 \pm \sqrt{1 - 4\times 2\times -2000}}{2\times 2} = 31.87 \text{ or } -31.37$

n must be positive, so $n = 31.87$.

If the sum has to be greater than 2000 then $n = 32$.

10 a Let a = first term and d = common difference.

Sum of the first two terms = 47

$\Rightarrow a + a + d = 47$

$\Rightarrow 2a + d = 47$

30th term = −62

Using nth term = $a + (n - 1)d$

$\Rightarrow a + 29d = -62$

(Note: $a + 12d$ is a common error here)

Our two simultaneous equations are

$$\begin{aligned} 2a + d &= 47 && \textbf{(1)} \\ a + 29d &= -62 && \textbf{(2)} \\ 2a + 58d &= -124 && \textbf{(3)} \quad ((\textbf{2}) \times 2) \\ 57d &= -171 && ((\textbf{3}) - (\textbf{1})) \\ d &= -3 && (\div 57) \end{aligned}$$

Substitute $d = -3$ into (**1**):

$2a - 3 = 47 \Rightarrow 2a = 50 \Rightarrow a = 25$

Therefore, first term = 25 and common difference = −3.

b using $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$$S_{60} = \frac{60}{2}\left(2a + (60-1)d\right) = 30(2a + 59d)$$

Substituting $a = 25$, $d = -3$ gives

$S_{60} = 30(2 \times 25 + 59 \times (-3))$

$= 30(50 - 177) = 30 \times (-127)$

$= -3810$

11 a Sum of integers divisible by 3 which lie between 1 and 400

$= 3 + 6 + 9 + 12 + \ldots + 399$

This is an arithmetic series with $a = 3$, $d = 3$ and $L = 399$.

Using $L = a + (n-1)d$

$$\begin{aligned} 399 &= 3 + (n-1) \times 3 \\ 399 &= 3 + 3n - 3 \\ 399 &= 3n \\ n &= 133 \end{aligned}$$

Therefore, there are 133 of these integers up to 400.

$$S_n = \frac{n}{2}(a + L) = \frac{133}{2}(3 + 399)$$

$$= \frac{133}{2} \times 402 = 26\,733$$

11 b Sum of integers not divisible by 3

$= 1 + 2 + 4 + 5 + 7 + 8 + 10 + \ldots + 400$

$$= \underbrace{(1 + 2 + 3 + 4 + \ldots + 399 + 400)}_{\text{Arithmetic series with } a=1, L=400, n=400}$$

$$- \underbrace{(3 + 6 + 9 + \ldots + 399)}_{\text{From part } \mathbf{a}\text{, this equals } 26\,733}$$

$$\begin{aligned} S_n &= \frac{400}{2}(1 + 400) \\ &= 200 \times 401 \\ &= 80\,200 \end{aligned}$$

So sum of integers not divisible by 3

$= 80\,200 - 26\,733$

$= 53\,467$

12 Let the shortest side be x.

$$S_{10} = \frac{10}{2}(x+2x) = 675$$
$$5(3x) = 675$$
$$15x = 675$$
$$x = 45$$

Length of shortest side is 45 cm.

13

$$\begin{array}{ccccccccc} \text{Sum} = & 4 & + & 8 & + & 12 & + \ldots + & 8n \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & \text{1st} & & \text{2nd} & & \text{3rd} & & 2n\text{th} \end{array}$$

This is an arithmetic series with $a = 4$, $d = 4$ and $n = 2n$.

Using $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$:

$$S_{2n} = \frac{\cancel{2}n}{\cancel{2}}\left(2\times 4 + (2n-1)\times 4\right)$$
$$= n(8+8n-4)$$
$$= n(8n+4)$$
$$= n\times 4(2n+1)$$
$$= 4n(2n+1)$$

14 a Replacing n with 1 $\Rightarrow U_2 = ku_1 - 4$

$u_1 = 2 \Rightarrow u_2 = 2k - 4$

Replacing n with 2 $\Rightarrow u_3 = ku_2 - 4$

$u_2 = 2k - 4 \Rightarrow u_3 = k(2k-4) - 4$

$\Rightarrow u_3 = 2k^2 - 4k - 4$

b Substitute $u_3 = 26$

$\Rightarrow \; 2k^2 - 4k - 4 = 26$

$\Rightarrow \; 2k^2 - 4k - 30 = 0 \qquad (\div 2)$

$\Rightarrow \; k^2 - 2k - 15 = 0 \qquad$ (factorise)

$\Rightarrow \; (k-5)(k+3) = 0$

$\Rightarrow \; k = 5, -3$

15 a Use nth term = $a + (n-1)d$:

5th term is 14 $\Rightarrow a + 4d = 14$

Use 1st term = a, 2nd term = $a + d$, 3rd term = $a + 2d$:

sum of first three terms = -3

$\Rightarrow \; a + a + d + a + 2d = -3$

$\Rightarrow \; 3a + 3d = -3 \quad (\div 3)$

$\Rightarrow \; a + d = -1$

Our simultaneous equations are

$a + 4d = 14 \qquad$ (**1**)

$a + d = -1 \qquad$ (**2**)

(**1**) − (**2**): $3d = 15 \qquad (\div 3)$

$d = 5$

Common difference = 5

Substitute $d = 5$ back into (**2**):

$a + 5 = -1$

$a = -6$

First term = -6

b nth term must be greater than 282

$\Rightarrow \; a + (n-1)d > 282$

$\Rightarrow \; -6 + 5(n-1) > 282 \quad (+6)$

$\Rightarrow \; 5(n-1) > 288 \quad (\div 5)$

$\Rightarrow \; (n-1) > 57.6 \quad (+1)$

$\Rightarrow \; n > 58.6$

$\therefore$ least value of $n = 59$

16 a We know nth term $= a + (n-1)d$

4th term is $3k$

$\Rightarrow \quad a + (4-1)d = 3k$

$\Rightarrow \quad a + 3d = 3k$

We know $S_n = \frac{n}{2}(2a + (n-1)d)$

Sum to 6 terms is $7k + 9$, therefore

$\frac{6}{2}(2a + (6-1)d) = 7k + 9$

$3(2a + 5d) = 7k + 9$

$6a + 15d = 7k + 9$

The simultaneous equations are

$a + 3d = 3k$ **(1)**

$6a + 15d = 7k + 9$ **(2)**

(1) × 5: $5a + 15d = 15k$ **(3)**

(2) − **(3)** : $1a = -8k + 9$

$\Rightarrow \quad a = 9 - 8k$

First term is $9 - 8k$.

b Substituting this in **(1)** gives

$$\begin{aligned} 9 - 8k + 3d &= 3k \\ 3d &= 11k - 9 \\ d &= \frac{11k-9}{3} \end{aligned}$$

Common difference is $\frac{11k-9}{3}$.

c If the 7th term is 12, then

$a + 6d = 12$

Substitute values of a and d:

$$\begin{aligned} -8k + 9 + 6 \times \left(\frac{11k-9}{3}\right) &= 12 \\ -8k + 9 + 2(11k - 9) &= 12 \\ -8k + 9 + 22k - 18 &= 12 \\ 14k - 9 &= 12 \\ 14k &= 21 \\ k &= \frac{21}{14} \\ &= 1.5 \end{aligned}$$

d Calculate values of a and d first:

$a = 9 - 8k = 9 - 8 \times 1.5 = 9 - 12 = -3$

$$d = \frac{11k-9}{3} = \frac{11 \times 1.5 - 9}{3} = \frac{16.5 - 9}{3} = \frac{7.5}{3}$$
$$= 2.5$$

$$\begin{aligned} S_{20} &= \frac{20}{2}(2a + (20-1)d) \\ &= 10(2a + 19d) \\ &= 10(2 \times (-3) + 19 \times 2.5) \\ &= 10(-6 + 47.5) \\ &= 10 \times 41.5 \\ &= 415 \end{aligned}$$

Sum to 20 terms is 415.

17 a $a_1 = p$

$a_2 = \frac{1}{p}$

$a_3 = \frac{1}{\frac{1}{p}} = 1 \times \frac{p}{1} = p$

$a_4 = \frac{1}{p}$

So the sequence is periodic with order 2.

17 b $\sum_{r=1}^{1000} a_n = \frac{1000}{2}\left(p+\frac{1}{p}\right)$

$= 500\left(p+\frac{1}{p}\right)$

18 a $a_1 = k$

$a_2 = 2k+6$

$a_3 = 2(2k+6)+6 = 4k+18$

As the sequence is increasing:

$a_1 < a_2 < a_3$

$k < 2k+6 < 4k+18$

$k > -6$

b $a_4 = 2(4k+18)+6 = 8k+42$

c $\sum_{r=1}^{4} a_r = k+2k+6+4k+18+8k+42$

$= 15k+66$

$= 3(5k+22)$

Therefore, $\sum_{r=1}^{4} a_r$ is divisible by 3.

19 a $a = 130$, $S_\infty = 650$

$\frac{130}{1-r} = 650$

$130 = 650 - 650r$

$-520 = -650r$

$r = \frac{4}{5}$

b $u_7 - u_8 = ar^6 - ar^7$

$= 130\left(\frac{4}{5}\right)^6 - 130\left(\frac{4}{5}\right)^7$

$= 6.82$

c $S_7 = \frac{130(1-0.8^7)}{1-0.8}$

$= 513.69$ (2 d.p.)

d $\frac{130(1-0.8^n)}{1-0.8} > 600$

$\frac{130(1-0.8^n)}{0.2} > 600$

$1-0.8^n > \frac{12}{13}$

$0.8^n < \frac{1}{13}$

$n\log 0.8 < -\log 13$

$n > \frac{-\log 13}{\log 0.8}$

20 a $a = 25\,000$, $r = 1.02$

$ar^2 = 25\,000 \times 1.02^2$

$= 26\,010$

b $25\,000 \times 1.02^n > 50\,000$

$1.02^n > 2$

$n \log 1.02 > \log 2$

$n > \frac{\log 2}{\log 1.02}$

c $n > 35.003$

Initial year was 2012, and n is an integer, so 2048.

d $S_8 = \frac{25\,000(1.02^8 - 1)}{1.02 - 1} = 214\,574.22$

$= 214\,574$

e People may visit the doctor more frequently than once a year, some may not visit at all. It depends on their state of health.

21 a 3, 5, 7, …

nth term $= (3 + (n-1)2) = 2n + 1$

b $2k + 1 = 301$

$k = 150$

c i $S_q = \frac{q}{2}(2\times 3+(q-1)2) = p$

$q(q+2) = p$

$q^2 + 2q = p$

$q^2 + 2q - p = 0$

21 c ii $p > 1520$

$q^2 + 2q = p$

$q^2 + 2q > 1520$

$q^2 + 2q - 1520 > 0$

$q^2 + 2q - 1520 = 0$

$(q-38)(q+40) = 0$

$q = 38$ or -40

As $q^2 + 2q - 1520 > 0$, $q > 38$

minimum numbers of rows is 39.

22 a $ar = -3,\ S_\infty = 6.75$

$a = -\frac{3}{r}$

$\frac{a}{1-r} = 6.75$

$-\frac{3}{r} \times \frac{1}{1-r} = 6.75$

$\frac{-3}{r-r^2} = 6.75$

$6.75r - 6.75r^2 + 3 = 0$

$27r^2 - 27r - 12 = 0$

b $9r^2 - 9r - 4 = 0$

$(3r-4)(3r+1) = 0$

$r = \frac{4}{3}$ or $r = -\frac{1}{3}$

As the series is convergent, $|r| < 1$ so

$r = -\frac{1}{3}$

22 c $ar = -3$ so $a = 9$

$$S_5 = \frac{9\left(1-\left(-\frac{1}{3}\right)^5\right)}{1+\frac{1}{3}}$$

$$= \frac{27}{4}\left(1-\left(-\frac{1}{3}\right)^5\right)$$

$$= 6.78$$

Challenge

a $u_{n+2} = 5u_{n+1} - 6u_n$

$= 5[p(3^{n+2}) + q(2^{n+2})] - 6[p(3^n) + q(2^n)]$

$= 5\left(p\left(\frac{1}{3}\right)(3^{n+2}) + q\left(\frac{1}{2}\right)(2^{n+2})\right)$

$\quad - 6\left(p\left(\frac{1}{3}\right)^2(3^{n+2}) + q\left(\frac{1}{2}\right)^2(2^{n+2})\right)$

$= \left(\frac{5}{3}p - \frac{6}{9}p\right)(3^{n+2}) + \left(\frac{5}{2}q - \frac{6}{4}q\right)(2^{n+2})$

$= p(3^{n+2}) + q(2^{n+2})$

b $u_1 = 5 = p(3^1) + q(2^1)$

$u_2 = 12 = p(3^2) + q(2^2)$

$5 = 3p + 2q$

$12 = 9p + 4q$

Solving simultaneously:

$10 = 6p + 4q$ **(1)**

$12 = 9p + 4q$ **(2)**

(2) − **(1)**:

$2 = 3p$

$p = \frac{2}{3}$

$2q = 5 - 2 = 3$

$q = \frac{3}{2}$

Therefore, $u_n = \left(\frac{2}{3}\right)3^n + \left(\frac{3}{2}\right)2^n$

c $u_{100} = \left(\frac{2}{3}\right)3^{100} + \left(\frac{3}{2}\right)2^{100}$

$= 3.436 \times 10^{47}$

So it contains 48 digits.