The *t*− formulae 5A

1 a $\tan\frac{\theta}{2} = t = \frac{2}{3}$, so

$$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}} = \frac{\frac{2}{3}}{\sqrt{1+\left(\frac{2}{3}\right)^2}}$$
$$= \frac{\frac{2}{3}}{\sqrt{1+\frac{4}{9}}} = \frac{\frac{2}{3}}{\sqrt{\frac{13}{9}}} = \frac{\frac{2}{3}}{\frac{\sqrt{13}}{3}} = \frac{2}{\sqrt{13}}$$

b $\sin\theta = \frac{2t}{1+t^2} = \frac{2\left(\frac{2}{3}\right)}{1+\left(\frac{2}{3}\right)^2} = \frac{\frac{4}{3}}{\frac{13}{9}} = \frac{36}{39} = \frac{12}{13}$

c $\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1-\left(\frac{2}{3}\right)^2}{1+\left(\frac{2}{3}\right)^2} = \frac{\frac{5}{9}}{\frac{13}{9}} = \frac{5}{13}$

d $\tan\theta = \frac{2t}{1-t^2} = \frac{2\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)^2} = \frac{\frac{4}{3}}{\frac{5}{9}} = \frac{36}{15} = \frac{12}{5}$

2 a $\tan\frac{\theta}{2} = t = 2$, so

$$\sin\theta = \frac{2t}{1+t^2} = \frac{2(2)}{1+2^2} = \frac{4}{5}$$

b $\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1-2^2}{1+2^2} = -\frac{3}{5}$

c $\tan\theta = \frac{2t}{1-t^2} = \frac{2(2)}{1-2^2} = -\frac{4}{3}$

d $\sec\theta + \cot\theta = \frac{1}{\cos\theta} + \frac{1}{\tan\theta}$

$$= \frac{1+t^2}{1-t^2} + \frac{1-t^2}{2t} = \frac{1+2^2}{1-2^2} + \frac{1-2^2}{2(2)} = -\frac{5}{3} - \frac{3}{4}$$
$$= -\frac{20}{12} - \frac{9}{12} = -\frac{29}{12}$$

3 a $\sin\frac{\theta}{2} = \frac{4}{5}$

So $\cos\frac{\theta}{2} = +\sqrt{1-\sin^2\frac{\theta}{2}} = +\sqrt{1-\left(\frac{4}{5}\right)^2}$

$$= +\sqrt{1-\frac{16}{25}} = +\sqrt{\frac{9}{25}} = \frac{3}{5}$$

Note: the positive root is taken, since $0 \leqslant \frac{\theta}{2} < \frac{\pi}{2}$ and $\cos\frac{\theta}{2}$ is positive in this range.

Also, $t = \tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$

Now $\sin\theta = \frac{2t}{1+t^2} = \frac{2\left(\frac{4}{3}\right)}{1+\left(\frac{4}{3}\right)^2}$

$$= \frac{\frac{8}{3}}{\frac{25}{9}} = \frac{72}{75} = \frac{24}{25}$$

b $\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1-\left(\frac{4}{3}\right)^2}{1+\left(\frac{4}{3}\right)^2} = \frac{-\frac{7}{9}}{\frac{25}{9}} = -\frac{7}{25}$

c $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(-\frac{7}{25}\right)} = -\frac{25}{7}$

d $\frac{\cos\theta}{\sin\theta(1+\cot\theta)} = \frac{\cos\theta}{\sin\theta + \sin\theta\cot\theta}$

$$= \frac{\cos\theta}{\sin\theta+\cos\theta} = \frac{-\frac{7}{25}}{\frac{24}{25}-\frac{7}{25}} = -\frac{7}{17}$$

4 a $\cos\frac{\theta}{2} = -\frac{5}{13}$

So $\sin\frac{\theta}{2} = \sqrt{1-\cos^2\frac{\theta}{2}} = \sqrt{1-\left(-\frac{5}{13}\right)^2}$

$= \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

Note: the positive root is taken, since $\frac{\pi}{2} \leqslant \frac{\theta}{2} < \pi$ and $\sin\frac{\theta}{2}$ is positive in this range.

Also $t = \tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\frac{12}{13}}{-\frac{5}{13}} = -\frac{12}{5}$

Now $\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1-\left(-\frac{12}{5}\right)^2}{1+\left(-\frac{12}{5}\right)^2}$

$= \frac{25-144}{25+144} = -\frac{119}{169}$

b $\tan^2\theta = \left(\frac{2t}{1-t^2}\right)^2 = \left(\frac{2\left(-\frac{12}{5}\right)}{1-\left(-\frac{12}{5}\right)^2}\right)^2$

$= \left(\frac{-\frac{24}{5}}{-\frac{119}{25}}\right)^2 = \left(\frac{600}{595}\right)^2 = \left(\frac{120}{119}\right)^2 = \frac{14\,400}{14\,161}$

c $\sec\theta + \operatorname{cosec}\theta = \frac{1}{\cos\theta} + \frac{1}{\sin\theta}$

$= \frac{1+t^2}{1-t^2} + \frac{1+t^2}{2t}$

$= \frac{1+\left(-\frac{12}{5}\right)^2}{1-\left(-\frac{12}{5}\right)^2} + \frac{1+\left(-\frac{12}{5}\right)^2}{2\left(-\frac{12}{5}\right)}$

$= \frac{\frac{169}{25}}{-\frac{119}{25}} + \frac{\frac{169}{25}}{-\frac{24}{5}} = -\frac{169}{119} - \frac{169}{120}$

$= \frac{-20\,280 - 20\,111}{14\,280} = -\frac{40\,391}{14\,280}$

d $\frac{\sec\theta}{\operatorname{cosec}\theta + \cos\theta} = \frac{-\frac{169}{119}}{-\frac{169}{120} + \left(-\frac{119}{169}\right)}$

$= \frac{-\frac{169}{119}}{-\frac{169}{120} - \frac{119}{169}} = \frac{-\frac{169}{119}}{-\frac{42\,841}{20\,280}} = \frac{3\,427\,320}{5\,098\,079}$

5 a $\operatorname{cosec}\frac{\theta}{2} = \frac{25}{24}$, so $\sin\frac{\theta}{2} = \frac{24}{25}$

Now $\cos\frac{\theta}{2} = -\sqrt{1-\sin^2\frac{\theta}{2}} = -\sqrt{1-\left(\frac{24}{25}\right)^2}$

$= -\sqrt{1-\frac{576}{625}} = -\sqrt{\frac{49}{625}} = -\frac{7}{25}$

Note: the negative root is taken, since $\frac{\pi}{2} \leqslant \frac{\theta}{2} < \pi$ and $\cos\frac{\theta}{2}$ is negative in this range.

Also $t = \tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\frac{24}{25}}{-\frac{7}{25}} = -\frac{24}{7}$

$\tan\theta = \frac{2t}{1-t^2} = \frac{2\left(-\frac{24}{7}\right)}{1-\left(-\frac{24}{7}\right)^2}$

$= \frac{-\frac{48}{7}}{1-\frac{576}{49}} = \frac{-\frac{48}{7}}{-\frac{527}{49}} = \frac{336}{527}$

b $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right) = \frac{4t\left(1-t^2\right)}{\left(1+t^2\right)^2}$

$= \frac{4\left(-\frac{24}{7}\right)\left(1-\left(-\frac{24}{7}\right)^2\right)}{\left(1+\left(-\frac{24}{7}\right)^2\right)^2} = \frac{\left(-\frac{96}{7}\right)\left(1-\frac{576}{49}\right)}{\left(1+\frac{576}{49}\right)^2}$

$= \frac{\left(-\frac{96}{7}\right)\left(-\frac{527}{49}\right)}{\left(\frac{625}{49}\right)^2} = \frac{\frac{50\,592}{343}}{\frac{390\,625}{2401}}$

$= \frac{121\,471\,392}{133\,984\,375} = \frac{354\,144}{390\,625}$

5 c $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{1-t^2}{1+t^2}\right)^2 - \left(\frac{2t}{1+t^2}\right)^2$$

$$= \left(\frac{1-\left(-\frac{24}{7}\right)^2}{1+\left(-\frac{24}{7}\right)^2}\right)^2 - \left(\frac{2\left(-\frac{24}{7}\right)}{1+\left(-\frac{24}{7}\right)^2}\right)^2$$

$$= \left(\frac{-\frac{527}{49}}{\frac{625}{49}}\right)^2 - \left(\frac{-\frac{48}{7}}{\frac{625}{49}}\right)^2$$

$$= \left(-\frac{527}{625}\right)^2 - \left(-\frac{336}{625}\right)^2$$

$$= \frac{277\,729}{390\,625} - \frac{112\,896}{390\,625}$$

$$= \frac{164\,833}{390\,625}$$

d $\cot 2\theta = \dfrac{\cos 2\theta}{\sin 2\theta} = \dfrac{164\,833}{390\,625} \times \dfrac{390\,625}{354\,144}$

$$= \frac{164\,833}{354\,144}$$

6 a $\cos^2\theta = 1 - \sin^2\theta$

$$= 1 - \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2$$

$$= 1 - \left(\frac{3+1-2\sqrt{3}}{8}\right) = 1 - \left(\frac{4-2\sqrt{3}}{8}\right)$$

$$= \frac{8-\left(4-2\sqrt{3}\right)}{8} = \frac{4+2\sqrt{3}}{8}$$

So $\cos\theta = \sqrt{\dfrac{4+2\sqrt{3}}{8}} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\tan\theta = \frac{\sqrt{3}-1}{2\sqrt{2}} \div \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

6 b $\sin 2\theta = \dfrac{2t}{1+t^2} = \dfrac{2\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)}{1+\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2}$

$$= \frac{\left(\frac{2\sqrt{3}-2}{\sqrt{3}+1}\right)}{1+\left(\frac{4-2\sqrt{3}}{4+2\sqrt{3}}\right)} = \frac{\left(\frac{2\sqrt{3}-2}{\sqrt{3}+1}\right)}{\left(\frac{4+2\sqrt{3}+4-2\sqrt{3}}{4+2\sqrt{3}}\right)}$$

$$= \frac{\left(\frac{2\sqrt{3}-2}{\sqrt{3}+1}\right)}{\left(\frac{8}{4+2\sqrt{3}}\right)} = \left(\frac{2\sqrt{3}-2}{\sqrt{3}+1}\right)\left(\frac{4+2\sqrt{3}}{8}\right)$$

$$= \frac{4+4\sqrt{3}}{8+8\sqrt{3}} = \frac{4\left(1+\sqrt{3}\right)}{8\left(1+\sqrt{3}\right)} = \frac{1}{2}$$

$$\cos 2\theta = \frac{1-t^2}{1+t^2} = \frac{1-\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2}{1+\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2}$$

$$= \frac{1-\left(\frac{4-2\sqrt{3}}{4+2\sqrt{3}}\right)}{1+\left(\frac{4-2\sqrt{3}}{4+2\sqrt{3}}\right)} = \frac{\left(\frac{4\sqrt{3}}{4+2\sqrt{3}}\right)}{\left(\frac{8}{4+2\sqrt{3}}\right)}$$

$$= \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

c $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}}$

So $2\theta = \dfrac{\pi}{6}$ (since $0 \leqslant \dfrac{\theta}{2} < \dfrac{\pi}{2}$)

Therefore $\theta = \dfrac{\pi}{12}$

7 a $\sin^2 x = 1 - \cos^2 x = 1 - \left(-\dfrac{\sqrt{2+\sqrt{2}}}{2}\right)^2$

$= 1 - \dfrac{2+\sqrt{2}}{4} = \dfrac{2-\sqrt{2}}{4}$

So $\sin x = +\sqrt{\dfrac{2-\sqrt{2}}{4}} = \dfrac{+\sqrt{2-\sqrt{2}}}{2}$

Note: the positive root is taken, since $\dfrac{\pi}{2} \leqslant x < \pi$ and $\sin x$ is positive in this range.

Using $\tan x = \dfrac{\sin x}{\cos x}$

$\tan x = \dfrac{\sqrt{2-\sqrt{2}}}{2} \div -\dfrac{\sqrt{2+\sqrt{2}}}{2}$

$= -\dfrac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$

b $\tan 2x = \dfrac{2t}{1-t^2} = \dfrac{-\dfrac{2\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}}{1-\left(\dfrac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)^2}$

$= \dfrac{-\dfrac{2\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}}{1-\dfrac{2-\sqrt{2}}{2+\sqrt{2}}} = \dfrac{-\dfrac{2\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}}{\dfrac{2\sqrt{2}}{2+\sqrt{2}}}$

$= -\dfrac{2\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} \times \dfrac{2+\sqrt{2}}{2\sqrt{2}}$

$= -\dfrac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} \times \dfrac{2+\sqrt{2}}{\sqrt{2}}$

$= -\dfrac{\sqrt{\left(2-\sqrt{2}\right)\left(2+\sqrt{2}\right)}}{\sqrt{2}} = -\dfrac{\sqrt{2}}{\sqrt{2}} = -1$

c $\tan 2x = -1$,

so $2x = \dfrac{7\pi}{4}$ (since $\dfrac{\pi}{2} \leqslant x < \pi$)

Therefore $x = \dfrac{7\pi}{8}$

8 a $t = \tan\dfrac{5\pi}{12}$

So $\sin\dfrac{5\pi}{6} = \dfrac{2t}{1+t^2}$

Also $\sin\dfrac{5\pi}{6} = \dfrac{1}{2}$

Therefore $\dfrac{2t}{1+t^2} = \dfrac{1}{2}$

$4t = 1 + t^2$

$t^2 - 4t + 1 = 0$

b $t = \tan\dfrac{5\pi}{12}$

So $\cos\dfrac{5\pi}{6} = \dfrac{1-t^2}{1+t^2}$

Also $\cos\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{2}$

Therefore $\dfrac{1-t^2}{1+t^2} = -\dfrac{\sqrt{3}}{2}$

$2 - 2t^2 = -\sqrt{3} - \sqrt{3}t^2$

$2t^2 - \sqrt{3}t^2 = 2 + \sqrt{3}$

$t^2\left(2-\sqrt{3}\right) = 2 + \sqrt{3}$

$t^2 = \dfrac{2+\sqrt{3}}{2-\sqrt{3}}$

c $t^2 = \left(\dfrac{2+\sqrt{3}}{2-\sqrt{3}}\right)\left(\dfrac{2+\sqrt{3}}{2+\sqrt{3}}\right) = 7 + 4\sqrt{3}$

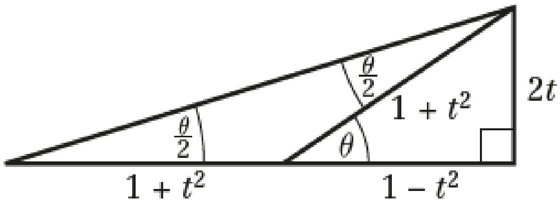
From part **a**), $t^2 = 4t - 1$

So $4t - 1 = 7 + 4\sqrt{3}$

$4t = 8 + 4\sqrt{3}$

$t = 2 + \sqrt{3}$

9 By considering angles and using Pythagoras' theorem, we can calculate



Hence, $\tan\frac{\theta}{2} = \frac{2t}{1+t^2+1-t^2} = t$

Also, by considering the smaller right-angled triangle we see

$\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\tan\theta = \frac{2t}{1-t^2}$