

Exam-style practice A Level

- 1 a The normal to the plane is given by the vector product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -7 & 0 \\ 0 & -8 & 4 \end{vmatrix} = -28\mathbf{i} - 8\mathbf{j} - 16\mathbf{k}$$

Hence after dividing the normal by a scalar the vector equation of the plane is

$$\mathbf{r} \cdot (7\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot (7\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$$

$$\mathbf{r} \cdot (7\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = -7 + 6 + 8 = 7$$

So the Cartesian equation is

$$7x + 2y + 4z = 7$$

- b The volume is given by the formula

$$\frac{1}{6} |\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})|$$

Which is given by the determinant

$$\frac{1}{6} \begin{vmatrix} -6 & -5 & 0 \\ 2 & -7 & 0 \\ 0 & -8 & 4 \end{vmatrix} = \frac{1}{6} (168 + 40) = \frac{104}{3}$$

- c The line \overline{ED} can be expressed in terms of unit normal to the plane \mathbf{n} as

$$(\overrightarrow{AD} \cdot \mathbf{n})\mathbf{n}$$

By drawing a picture we can see that

$$\sin \theta = \frac{|\overrightarrow{ED}|}{|\overrightarrow{CD}|} = \frac{|\overrightarrow{AD} \cdot \mathbf{n}|}{|\overrightarrow{CD}|}$$

We have

$$|\overrightarrow{CD}| = \sqrt{36 + 9 + 16} = \sqrt{61}$$

and

$$|\overrightarrow{AD} \cdot \mathbf{n}| = \frac{|-42 - 10|}{\sqrt{49 + 4 + 16}} = \frac{52}{\sqrt{69}}$$

Hence

$$\sin \theta = \frac{52}{\sqrt{61}\sqrt{69}}$$

So we have $\theta = 0.930$

- 2 a We have the following tableau of values

x_i	0.5	0.75	1	1.25	1.5
y_i	0.390	0.511	0.678	0.867	0.993

Hence using Simpson's rule our approximation is

$$\int_{0.5}^{1.5} f(x) dx \approx \frac{1}{3} \times 0.25(f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)) = 0.68778 \text{ (5dp)}$$

2 b To solve the integral exactly we use the substitution $t = \tan \frac{x}{2}$ we have

$$\frac{dt}{dx} = \frac{1}{2} \sec^2 \left(\frac{x}{2} \right) = \frac{1}{2} (1+t^2)$$

So the transformed integral becomes

$$\int_{\tan \frac{1}{4}}^{\tan \frac{3}{4}} \frac{2}{4-3 \sin x} (1+t^2)^{-1} dt$$

Now recall that $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

and hence

$$\sin x = 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} = \frac{2t}{1+t^2}$$

So the transformed integral becomes

$$\int_{\tan \frac{1}{4}}^{\tan \frac{3}{4}} \frac{2(1+t^2)^{-1}}{4-6t(1+t^2)^{-1}} dt = \int_{\tan \frac{1}{4}}^{\tan \frac{3}{4}} \frac{1}{2t^2-3t+2} dt$$

Now complete the square to get

$$\int_{\tan \frac{1}{4}}^{\tan \frac{3}{4}} \frac{1}{\left(\frac{4t-3}{2\sqrt{2}} \right)^2 + \frac{7}{8}} dt = \int_{\tan \frac{1}{4}}^{\tan \frac{3}{4}} \frac{8}{(4t-3)^2 + 7} dt$$

Now use the substitution $u = \frac{4t-3}{\sqrt{7}}$ so the integral becomes

$$\int_{\frac{4 \tan \frac{1}{4} - 3}{\sqrt{7}}}^{\frac{4 \tan \frac{3}{4} - 3}{\sqrt{7}}} \frac{8}{7u^2 + 7} \frac{\sqrt{7}}{4} du = \frac{2}{\sqrt{7}} \int_{\frac{4 \tan \frac{1}{4} - 3}{\sqrt{7}}}^{\frac{4 \tan \frac{3}{4} - 3}{\sqrt{7}}} \frac{1}{u^2 + 1} du$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan \frac{3}{4} - 3}{\sqrt{7}} \right) - \frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan \frac{1}{4} - 3}{\sqrt{7}} \right)$$

$$= 0.68795$$

c The percentage error is given by

$$\frac{|0.68795 - 0.68778|}{0.68795} = 0.02\%$$

The approximation could be improved by taking more intervals

3 a We have to solve the differential equation

$$xy \frac{dy}{dx} + 3x^2 + y^2 = 0$$

Let us use the substitution $y = zx$ so that by the product rule

$$\frac{dy}{dx} = z + x \frac{dz}{dx}$$

So the differential equation becomes

$$x^2 z \left(z + x \frac{dz}{dx} \right) + 3x^2 + z^2 x^2 = 0$$

Cancelling common factors and simplifying leads to

$$zx \frac{dz}{dx} + 3 + 2z^2 = 0$$

Which can be written as

$$x \frac{dz}{dx} + \frac{3+2z^2}{z} = 0$$

b We have

$$\frac{z}{3+2z^2} \frac{dz}{dx} = -\frac{1}{x}$$

Integrating both sides with respect to x yields

$$\int \frac{z}{3+2z^2} dz = \int -\frac{1}{x} dx$$

Hence

$$\frac{1}{4} \ln(3+2z^2) = -\ln(x) + C$$

Hence

$$(3+2z^2)^{\frac{1}{4}} = \frac{A}{x}$$

For some positive constant A

Raising to the power of 4 gives

$$3+2z^2 = \frac{A}{x^4}$$

Which simplifies to

$$3x^4 + 2y^2x^2 = A$$

Now we know that when $x = 1$

we have $y = 5$ so A satisfies

$$A = 3 + 50 = 53$$

Hence the full solution is

$$3x^4 + 2y^2x^2 = 53$$

3 c When $y = 0$ we have

$$3x^4 = 53$$

So

$$x = \sqrt[4]{\frac{53}{3}} = 2.0502$$

Hence the cliff is 205.02m tall.

d When x is small the model is approximated by

$$2x^2y^2 = 53$$

Which suggests the velocity must be very large when x is small which is unrealistic as the velocity should be small after just jumping.

4 a As $x \rightarrow 1$ we have

$$5x^4 - 3x^2 - 1 \rightarrow 1$$

but

$$11 - 2x - 9x^3 \rightarrow 0$$

Hence we are not in a situation where L'Hopital's rule applies.

b Now both terms in the expression tend to zero as $x \rightarrow 1$ so we are in a position where we can use L'Hopital's rule which gives

$$\lim_{x \rightarrow 1} \frac{5x^4 - 3x^2 - 2}{11 - 2x - 9x^3} = \lim_{x \rightarrow 1} \frac{20x^3 - 6x}{-2 - 27x^2}$$

And one may evaluate the second limit to get

$$\lim_{x \rightarrow 1} \frac{20x^3 - 6x}{-2 - 27x^2} = -\frac{14}{29}$$

5 a If the line intersects the hyperbola then

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and

$$y = mx + c$$

Hence

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

So

$$b^2x^2 - a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

Rearranging gives

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0$$

Now the line is a tangent if and only if it intersects in exactly one point, hence the above quadratic must have one root so by considering the discriminant we have

$$4a^4m^2c^2 + 4a^2(b^2 - a^2m^2)(b^2 + c^2) = 0$$

Hence

$$a^2m^2c^2 + (b^2 - a^2m^2)(b^2 + c^2) = 0$$

So

$$a^2m^2c^2 = -b^4 - b^2c^2 + a^2m^2b^2 + a^2m^2c^2$$

So

$$b^4 + b^2c^2 = a^2m^2b^2$$

Hence the result

$$b^2 + c^2 = a^2m^2$$

5 b We now consider the hyperbola

$$\frac{x^2}{26} - \frac{y^2}{25} = 1$$

And we want to find the tangent that passes through $(2, 3)$ using the result from the previous part we must have

$$26m^2 = 25 + c^2$$

And since the tangent passes through $(2, 3)$ we must have

$$3 = 2m + c$$

Hence combining we have

$$26m^2 = 25 + (3 - 2m)^2$$

So

$$26m^2 = 25 + 9 - 12m + 4m^2$$

$$22m^2 + 12m - 34 = 0$$

$$11m^2 + 6m - 17 = 0$$

Which we can solve to give the solutions

$$m = 1$$

and

$$m = -\frac{17}{11}$$

Hence there are two tangents that pass through the prescribed point given by

$$y = x + 1$$

and

$$y = -\frac{17}{11}x + \frac{67}{11}$$

6 We want to find a series solution of

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0$$

Under the initial conditions at $x = 0$

$$y = \frac{dy}{dx} = 1$$

Using the differential equation at $x = 0$ we must have

$$\frac{d^2y}{dx^2} + 3 = 0$$

Hence the coefficient of x^2 in the series solution is $-\frac{3}{2}$, differentiating the

differential equation gives

$$\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2\frac{dy}{dx} = 0$$

Evaluating at $x = 0$ leads to

$$\frac{d^3y}{dx^3} - 6 + 2 = 0$$

Hence the coefficient of x^3 is $\frac{4}{3!}$

So the series solution up to the x^3 term is

$$y = 1 + x - \frac{3}{2}x^2 + \frac{2}{3}x^3$$

7 We wish to solve the inequality

$$\left| \frac{x}{x+3} \right| < 2-x$$

First of all note that when $x > 2$ the right hand side is negative and so the inequality can never be satisfied.

Now consider the first case where the expression inside the modulus is positive, i.e.

$$\frac{x}{x+3} > 0$$

and note that this is satisfied when either $x > 0$ or $x < -3$ if $x > 0$ then the inequality becomes

$$\frac{x}{x+3} < 2-x$$

Hence

$$x < (2-x)(x+3)$$

So

$$x^2 + 2x - 6 < 0$$

By finding the roots one sees that this quadratic is negative if and only if

$$-1 - \sqrt{7} < x < \sqrt{7} - 1$$

But we are in the case where $x > 0$ hence in this case the inequality is satisfied if

$$0 < x < \sqrt{7} - 1$$

Now consider the case $x < -3$ then the inequality is

$$\frac{x}{x+3} < 2-x$$

Which becomes

$$x > (2-x)(x+3)$$

Hence

$$x^2 + 2x - 6 > 0$$

Which is satisfied when $x < 1 - \sqrt{7}$

Finally consider the case where the expression inside the modulus is negative in which case we have $-3 < x < 0$ so the inequality becomes

$$-\frac{x}{x+3} < 2-x$$

So

$$\frac{x}{x+3} > x-2$$

So

$$x > (x-2)(x+3)$$

Hence

$$x > x^2 + x - 6$$

Hence

$$x^2 < 6 \text{ so } x > -\sqrt{6}$$

Hence the solution set from this case is

$-\sqrt{6} < x < 0$ putting all the cases together gives the solution set

$$\{x : x < -1 - \sqrt{7}\} \cup \{x : -\sqrt{6} < x < \sqrt{7} - 1\}$$

8 We have $y = e^x \sin x$ we can write

$y = uv$ where $u = e^x$ and

$v = \sin x$ then Leibnitz's theorem gives

$$\frac{d^6 y}{dx^6} = \sum_{k=0}^6 \binom{6}{k} \frac{d^{6-k} u}{dx^{6-k}} \frac{d^k v}{dx^k}$$

Note that

$$\frac{d^k u}{dx^k} = u = e^x \text{ for all } k \text{ then the sum becomes}$$

$$\begin{aligned} \frac{d^6 y}{dx^6} &= e^x (-\sin x + 6 \cos x + 15 \sin x - 20 \cos x \\ &\quad - 15 \sin x + 6 \cos x + \sin x) \\ &= -8e^x \cos x \end{aligned}$$

And the usual product rule gives

$$\frac{dy}{dx} = e^x (\sin x + \cos x)$$

Hence

$$\begin{aligned} \frac{d^6 y}{dx^6} + 8 \frac{dy}{dx} - 8y &= -8e^x \cos x + 8e^x (\sin x + \cos x) - 8e^x \sin x \\ &= 0 \end{aligned}$$

As required.