

## Series ME 3

$$1 \text{ a } \sum_{r=1}^{10} r = \frac{1}{2} \times 10 \times 11 = 55$$

$$\begin{aligned} 1 \text{ b } \sum_{r=10}^{50} r &= \sum_{r=1}^{50} r - \sum_{r=1}^9 r \\ &= \frac{1}{2} \times 50 \times 51 - \frac{1}{2} \times 9 \times 10 \\ &= 1230 \end{aligned}$$

$$1 \text{ c } \sum_{r=1}^{10} r^2 = \frac{1}{6} \times 10 \times 11 \times 21 = 385$$

$$1 \text{ d } \sum_{r=1}^{10} r^3 = \frac{1}{4} \times 10^2 \times 11^2 = 3025$$

$$\begin{aligned} 1 \text{ e } \sum_{r=26}^{50} r^2 &= \sum_{r=1}^{50} r^2 - \sum_{r=1}^{25} r^2 \\ &= \frac{1}{6} \times 50 \times 51 \times 101 - \frac{1}{6} \times 25 \times 26 \times 51 \\ &= 37\,400 \end{aligned}$$

$$\begin{aligned} 1 \text{ f } \sum_{r=50}^{100} r^3 &= \sum_{r=1}^{100} r^3 - \sum_{r=1}^{49} r^3 \\ &= \frac{1}{4} \times 100^2 \times 101^2 - \frac{1}{4} \times 49^2 \times 50^2 \\ &= 24\,001\,875 \end{aligned}$$

$$\begin{aligned} 1 \text{ g } \sum_{r=1}^{60} r + \sum_{r=1}^{60} r^2 \\ &= \frac{1}{2} \times 60 \times 61 + \frac{1}{6} \times 60 \times 61 \times 121 \\ &= 75\,640 \end{aligned}$$

$$\begin{aligned} 2 \text{ a } \sum_{r=1}^n (3r-5) &= 3 \sum_{r=1}^n r - \sum_{r=1}^n 5 \\ &= \frac{3}{2} n(n+1) - 5n \\ &= \frac{3}{2} n^2 + \frac{3}{2} n - 5n \\ &= \frac{3}{2} n^2 - \frac{7}{2} n \end{aligned}$$

$$\begin{aligned} 2 \text{ b } \sum_{r=1}^n (r^2 + r) &= \sum_{r=1}^n r^2 + \sum_{r=1}^n r \\ &= \frac{1}{6} n(n+1)(2n+1) + \frac{1}{2} n(n+1) \\ &= \frac{1}{6} n(n+1)(2n+1+3) \\ &= \frac{1}{6} n(n+1)(2n+4) \\ &= \frac{1}{3} n(n+1)(n+2) \end{aligned}$$

$$\begin{aligned} \text{c } \sum_{r=1}^n (3r^2 + 7r) &= 3 \sum_{r=1}^n r^2 + 7 \sum_{r=1}^n r \\ &= \frac{3}{6} n(n+1)(2n+1) + \frac{7}{2} n(n+1) \\ &= \frac{1}{6} n(n+1)(6n+3+21) \\ &= \frac{1}{6} n(n+1)(6n+24) \\ &= n(n+1)(n+4) \end{aligned}$$

$$\begin{aligned} \text{d } \sum_{r=1}^n (4r^3 + 6r^2) &= 4 \sum_{r=1}^n r^3 + 6 \sum_{r=1}^n r^2 \\ &= n^2(n+1)^2 + n(n+1)(2n+1) \\ &= n(n+1)(n^2 + n + 2n + 1) \\ &= n(n+1)(n^2 + 3n + 1) \end{aligned}$$

$$\begin{aligned} \text{e } \sum_{r=1}^n (r^2 - 2r) &= \sum_{r=1}^n r^2 + 2 \sum_{r=1}^n r \\ &= \frac{1}{6} n(n+1)(2n+1) - n(n+1) \\ &= \frac{1}{6} n(n+1)(2n+1-6) \\ &= \frac{1}{6} n(n+1)(2n-5) \end{aligned}$$

$$\begin{aligned} \text{f } \sum_{r=1}^n (r^2 - 3r) &= \sum_{r=1}^n r^2 + 3 \sum_{r=1}^n r \\ &= \frac{1}{6} n(n+1)(2n+1) - \frac{3}{2} n(n+1) \\ &= \frac{1}{6} n(n+1)(2n+1-9) \\ &= \frac{1}{6} n(n+1)(2n-8) \\ &= \frac{1}{3} n(n+1)(n-4) \end{aligned}$$

$$\begin{aligned}
 2 \text{ g } \sum_{r=1}^n (r^2 - 5) &= \sum_{r=1}^n r^2 - 5 \sum_{r=1}^n 1 \\
 &= \frac{1}{6}n(n+1)(2n+1) - 5n \\
 &= \frac{1}{6}n(2n^2 + 3n + 1 - 30) \\
 &= \frac{1}{6}n(2n^2 + 3n - 29)
 \end{aligned}$$

$$\begin{aligned}
 \text{h } \sum_{r=1}^n (2r^3 + 3r^2 + r + 4) &= 2 \sum_{r=1}^n r^3 + 3 \sum_{r=1}^n r^2 + \sum_{r=1}^n r + 4 \sum_{r=1}^n 1 \\
 &= \frac{1}{2}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) \\
 &\quad + \frac{1}{2}n(n+1) + 4n \\
 &= \frac{1}{2}n(n^3 + 2n^2 + n + 2n^2 + 3n + 1 + n + 1 + 8) \\
 &= \frac{1}{2}n(n^3 + 4n^2 + 5n + 10)
 \end{aligned}$$

$$\begin{aligned}
 3 \sum_{r=1}^{30} r(3r-1) &= \sum_{r=1}^{30} 3r^2 - r \\
 &= 3 \sum_{r=1}^{30} r^2 - \sum_{r=1}^{30} r \\
 &= \frac{1}{2} \times 30 \times 31 \times 61 - \frac{1}{2} \times 30 \times 31 \\
 &= 27\,900
 \end{aligned}$$

$$\begin{aligned}
 4 \text{ a } \sum_{r=1}^n r^2(r-3) &= \sum_{r=1}^n r^3 - 3r^2 \\
 &= \sum_{r=1}^n r^3 - 3 \sum_{r=1}^n r^2 \\
 &= \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)(2n+1) \\
 &= \frac{1}{4}n(n+1)(n(n+1) - 2(2n+1)) \\
 &= \frac{1}{4}n(n+1)(n^2 - 3n - 2) \\
 \text{so } a &= -3, b = -2.
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \sum_{r=1}^{20} r^2(r-3) &= \frac{1}{4} \times 20 \times 21 \times (20^2 - 60 - 2) \\
 &= 35\,490
 \end{aligned}$$

$$\begin{aligned}
 5 \text{ a } \sum_{r=1}^n (2r-1)^2 &= \sum_{r=1}^n 4r^2 - 4r + 1 \\
 &= 4 \sum_{r=1}^n r^2 - 4 \sum_{r=1}^n r + \sum_{r=1}^n 1 \\
 &= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n \\
 &= \frac{1}{3}n(n+1)(2(2n+1) - 6) + n \\
 &= \frac{1}{3}n(n+1)(4n-4) + n \\
 &= \frac{1}{3}n(4n^2 - 4 + 3) \\
 &= \frac{1}{3}n(2n-1)(2n+1)
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \sum_{r=1}^{2n} (2r-1)^2 &= \frac{1}{3} \times 2n(4n-1)(4n+1) \\
 &= \frac{2}{3}n(4n-1)(4n+1)
 \end{aligned}$$

$$\begin{aligned}
 6 \text{ a } \sum_{r=1}^n r(r+2) &= \sum_{r=1}^n r^2 + 2r \\
 &= \sum_{r=1}^n r^2 + 2 \sum_{r=1}^n r \\
 &= \frac{1}{6}n(n+1)(2n+1) + n(n+1) \\
 &= \frac{1}{6}n(n+1)(2n+7)
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \sum_{r=15}^{30} r(r+2) &= \sum_{r=1}^{30} r(r+2) - \sum_{r=1}^{14} r(r+2) \\
 &= \frac{1}{6} \times 30 \times 31 \times 67 - \frac{1}{6} \times 14 \times 15 \times 35 \\
 &= 9\,160
 \end{aligned}$$

$$\begin{aligned}
 7 \text{ a } \sum_{r=n+1}^{2n} r^2 &= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^n r^2 \\
 &= \frac{1}{6}(2n)(2n+1)(2(2n)+1) - \frac{1}{6}n(n+1)(2n+1) \\
 &= \frac{1}{6}n(2n+1)(2(4n+1) - n - 1) \\
 &= \frac{1}{6}n(2n+1)(7n+1)
 \end{aligned}$$

$$\text{b } \sum_{r=16}^{30} r^2 = \frac{1}{6} \times 15 \times 31 \times 106 = 8\,215$$

$$\begin{aligned}
 8 \text{ a } \sum_{r=1}^n (r^2 - r - 1) &= \sum_{r=1}^n r^2 - \sum_{r=1}^n r - \sum_{r=1}^n 1 \\
 &= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - n \\
 &= \frac{1}{3}n(n^2 - 4)
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ b } \sum_{r=10}^{40} (r^2 - r - 1) \\
 &= \sum_{r=1}^{40} (r^2 - r - 1) - \sum_{r=1}^9 (r^2 - r - 1) \\
 &= \frac{1}{3} \times 40 \times (40^2 - 4) - \frac{1}{3} \times 9 \times (9^2 - 4) \\
 &= 21049
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ c } \sum_{r=1}^n (r^2 - r - 1) &= \sum_{r=1}^{2n} r \\
 \Rightarrow \frac{1}{3}n(n^2 - 4) &= \frac{1}{2} \times 2n \times (2n + 1) \\
 \Rightarrow \frac{1}{3}n(n^2 - 4) &= n(2n + 1) \\
 \Rightarrow \frac{1}{3}(n^2 - 4) &= 2n + 1 \\
 \Rightarrow n^2 - 4 &= 6n + 3 \\
 \Rightarrow n^2 - 6n - 7 &= 0 \\
 \Rightarrow (n - 7)(n + 1) &= 0 \\
 \Rightarrow n &= 7
 \end{aligned}$$

$$\begin{aligned}
 9 \text{ a } \sum_{r=1}^n r(2r^2 + 1) &= \sum_{r=1}^n 2r^3 + r \\
 &= 2 \sum_{r=1}^n r^3 + \sum_{r=1}^n r \\
 &= \frac{1}{2}n^2(n+1)^2 + \frac{1}{2}n(n+1) \\
 &= \frac{1}{2}n(n+1)(n(n+1) + 1) \\
 &= \frac{1}{2}n(n+1)(n^2 + n + 1)
 \end{aligned}$$

$$\begin{aligned}
 9 \text{ b } \sum_{r=1}^n (100r^2 - r) &= 100 \sum_{r=1}^n r^2 - \sum_{r=1}^n r \\
 &= \frac{1}{6}n(n+1)(200n + 97)
 \end{aligned}$$

Now if  $\sum_{r=1}^n r(2r^2 + 1) = \sum_{r=1}^n (100r^2 - r)$ , then

$$\frac{1}{2}n(n+1)(n^2 + n + 1) = \frac{1}{6}n(n+1)(200n + 97)$$

$$\frac{1}{6}n(n+1)(3(n^2 + n + 1) - (200n + 97)) = 0$$

$$\frac{1}{6}n(n+1)(3n^2 - 197n - 94) = 0$$

But  $n \neq 0$ ,  $n \neq -1$  and  $3n^2 - 197n - 94$  has discriminant 39937 which is not square.

$$\begin{aligned}
 10 \text{ a } \sum_{r=1}^n r(r+1)^2 &= \sum_{r=1}^n r^3 + 2r^2 + r \\
 &= \sum_{r=1}^n r^3 + 2 \sum_{r=1}^n r^2 + \sum_{r=1}^n r \\
 &= \frac{1}{4}n^2(n+1)^2 + \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\
 &= \frac{1}{12}n(n+1)(n+2)(3n+5)
 \end{aligned}$$

$$\begin{aligned}
 10 \text{ b } \sum_{r=1}^n 70r &= 35n(n+1) \\
 \Rightarrow \frac{1}{12}n(n+1)(n+2)(3n+5) &= 35n(n+1) \\
 \Rightarrow \frac{1}{12}(n+2)(3n+5) &= 35 \\
 \Rightarrow 3n^2 + 11n + 10 &= 420 \\
 \Rightarrow 3n^2 + 11n - 410 &= 0 \\
 \Rightarrow (3n+41)(n-10) &= 0 \\
 \Rightarrow n &= 10
 \end{aligned}$$

$$\begin{aligned}
 11 \frac{1}{6}n(n+1)(2n+1) &= \frac{9}{2}(n+1)(n+2) + n + 1 \\
 \Rightarrow n(n+1)(2n+1) &= 27(n+1)(n+2) + 6(n+1) \\
 \Rightarrow (n+1)(2n^2 + n) &= (n+1)(27n + 54 + 6) \\
 \Rightarrow 2n^2 + n &= 27n + 60 \\
 \Rightarrow 2n^2 - 26n - 60 &= 0 \\
 \Rightarrow n^2 - 13n - 30 &= 0 \\
 \Rightarrow (n-15)(n+2) &= 0 \\
 \Rightarrow n &= 15
 \end{aligned}$$

## Challenge

$$\begin{aligned}
 \text{a } \sum_{i=1}^n \left( \sum_{r=1}^i r^2 \right) &= \sum_{i=1}^n \frac{1}{6} i(i+1)(2i+1) \\
 &= \frac{1}{6} \sum_{i=1}^n 2i^3 + 3i^2 + i \\
 &= \frac{1}{6} \left( \frac{1}{2} n^2(n+1)^2 + \frac{1}{2} n(n+1)(2n+1) + \frac{1}{2} n(n+1) \right) \\
 &= \frac{1}{12} (n(n+1)(n^2 + n + 2n + 1 + 1)) \\
 &= \frac{1}{12} n(n+1)(n^2 + 3n + 2) \\
 &= \frac{1}{12} n(n+1)(n+2)(n+1) \\
 &= \frac{1}{12} n(n+1)^2(n+2)
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \sum_{j=1}^n \left( \sum_{i=1}^j \left( \sum_{r=1}^i r \right) \right) &= \sum_{j=1}^n \left( \frac{1}{2} \sum_{i=1}^j i(i+1) \right) \\
 &= \frac{1}{2} \sum_{j=1}^n \left( \sum_{i=1}^j i^2 \right) + \frac{1}{2} \sum_{j=1}^n \left( \sum_{i=1}^j i \right) \\
 &= \frac{1}{24} n(n+1)^2(n+2) + \frac{1}{4} \sum_{j=1}^n j^2 + \frac{1}{4} \sum_{j=1}^n j \\
 &= \frac{1}{24} n(n+1)^2(n+2) + \frac{1}{24} n(n+1)(2n+1) \\
 &\quad + \frac{1}{8} n(n+1) \\
 &= \frac{1}{24} n(n+1)(n^2 + 5n + 6) \\
 &= \frac{1}{24} n(n+1)(n+2)(n+3)
 \end{aligned}$$