

Complex numbers 1F

1 a $f(z) = z^3 - 6z^2 + 21z - 26$

$$\begin{aligned} f(2) &= (2)^3 - 6(2)^2 + 21(2) - 26 \\ &= 8 - 24 + 42 - 26 \\ &= 0 \end{aligned}$$

b If $f(2) = 0$ then $z - 2$ is a factor of $f(z)$

$$\begin{array}{r} z^2 - 4z + 13 \\ z-2 \overline{) z^3 - 6z^2 + 21z - 26} \\ \underline{z^3 - 2z^2} \\ -4z^2 + 21z \\ \underline{-4z^2 + 8z} \\ 13z - 26 \\ \underline{13z - 26} \\ 0 \end{array}$$

So $0 = f(z)$

$$\begin{aligned} &= z^3 - 6z^2 + 21z - 26 \\ &= (z - 2)(z^2 - 4z + 13) \end{aligned}$$

Either $(z - 2) = 0$ or $(z^2 - 4z + 13) = 0$

Solve $z^2 - 4z + 13 = 0$ by completing the square:

$$\begin{aligned} (z - 2)^2 - 4 + 13 &= 0 \\ (z - 2)^2 &= -9 \\ z - 2 &= \pm 3i \\ z &= 2 \pm 3i \end{aligned}$$

So the roots of $z^3 - 6z^2 + 21z - 26 = 0$ are
2, $2 + 3i$ and $2 - 3i$

2 a $f(z) = 2z^3 + 5z^2 + 9z - 6$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) - 6 \\ &= \frac{2}{8} + \frac{5}{4} + \frac{9}{2} - 6 \\ &= \frac{1}{4} + \frac{5}{4} + \frac{18}{4} - \frac{24}{4} \\ &= 0 \end{aligned}$$

2 b $f\left(\frac{1}{2}\right) = 0 \Rightarrow 2z - 1$ is a factor of $f(z) = 0$

Hence

$$\begin{aligned} f(z) &= 2z^3 + 5z^2 + 9z - 6 \\ &= (2z - 1)(z^2 + bz + c) \\ &= 2z^3 + (2b - 1)z^2 + (2c - b)z - c \end{aligned}$$

Equating z^2 terms:

$$2b - 1 = 5, \text{ so } b = 3$$

Equating constant terms:

$$-c = -6, \text{ so } c = 6$$

Hence $f(z) = (2z - 1)(z^2 + 3z + 6)$

c $f(z) = 0 \Rightarrow$ Either $(2z - 1) = 0$

or $(z^2 + 3z + 6) = 0$

Solve $(z^2 + 3z + 6) = 0$ by completing the square:

$$\left(z + \frac{3}{2}\right)^2 - \frac{9}{4} + 6 = 0$$

$$\left(z + \frac{3}{2}\right)^2 = -\frac{15}{4}$$

$$z + \frac{3}{2} = \pm \frac{\sqrt{15}}{2}i$$

$$z = -\frac{3}{2} \pm \frac{\sqrt{15}}{2}i$$

So the roots of $2z^3 + 5z^2 + 9z - 6 = 0$

are $\frac{1}{2}$, $\left(-\frac{3}{2} + \frac{\sqrt{15}}{2}i\right)$, and $\left(-\frac{3}{2} - \frac{\sqrt{15}}{2}i\right)$

$$3 \quad g(z) = 2z^3 - 4z^2 - 5z - 3$$

$z = 3$ is a root of $g(z) = 0$

$\Rightarrow z - 3$ is a factor of $g(z)$

$$\begin{array}{r} 2z^2 + 2z + 1 \\ z-3 \overline{) 2z^3 - 4z^2 - 5z - 3} \\ \underline{2z^3 - 6z^2} \\ 2z^2 - 5z \\ \underline{2z^2 - 6z} \\ z - 3 \\ \underline{z - 3} \\ 0 \end{array}$$

$$\text{So } g(z) = (z-3)(2z^2 + 2z + 1) = 0$$

Solve $2z^2 + 2z + 1 = 0$ using the quadratic formula:

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{(2)^2 - (4)(2)(1)}}{2(2)} \\ z &= \frac{-2 \pm \sqrt{4-8}}{4} \\ z &= \frac{-2 \pm 2i}{4} \\ z &= -\frac{1}{2} \pm \frac{1}{2}i \end{aligned}$$

So the roots of $2z^3 - 4z^2 - 5z - 3 = 0$

are 3 , $-\frac{1}{2} + \frac{1}{2}i$ and $-\frac{1}{2} - \frac{1}{2}i$

$$4 \quad \text{a} \quad p(z) = z^3 + 4z^2 - 15z - 68$$

$z = -4 + i$ is a solution to $p(z) = 0$, so

$z = -4 - i$ is also a solution to $p(z) = 0$.

Hence $(z - (-4 + i))(z - (-4 - i))$ is a factor of $p(z)$

$$\begin{aligned} &(z - (-4 + i))(z - (-4 - i)) \\ &= z^2 - (-4 - i)z - (-4 + i)z \\ &\quad + (-4 + i)(-4 - i) \\ &= z^2 + 4z + iz + 4z - iz + 16 + 4i - 4i - i^2 \\ &= z^2 + 8z + 17 \\ \therefore z^2 + 8z + 17 &\text{ is a factor of } p(z) \end{aligned}$$

$$4 \quad \text{b} \quad 0 = p(z)$$

$$= z^3 + 4z^2 - 15z - 68$$

$$= (z^2 + 8z + 17)(az + b)$$

$$= az^3 + (b + 8a)z^2 + (17a + 8b)z + 17b$$

Equate z^3 coefficients:

$$a = 1$$

Equate constants:

$$17b = -68 \Rightarrow b = -4$$

Hence $z - 4$ is a factor of $p(z)$ and $z = 4$ is a solution of $p(z) = 0$

So the roots of $z^3 + 4z^2 - 15z - 68 = 0$ are 4 , $-4 + i$ and $-4 - i$

$$5 \quad \text{a} \quad z^3 + 9z^2 + 33z + 25 = (z+1)(z^2 + az + b)$$

$$= z^3 + (1+a)z^2 + (a+b)z + b$$

Equate z^2 terms:

$$a + 1 = 9, \text{ so } a = 8$$

Equate constant terms:

$$b = 25$$

$$\text{b} \quad f(z) = (z+1)(z^2 + 8z + 25) = 0$$

Either $z + 1 = 0 \Rightarrow z = -1$ **or**

$$z^2 + 8z + 25 = 0$$

Solve $z^2 + 8z + 25 = 0$ by completing the square:

$$(z + 4)^2 - 16 + 25 = 0$$

$$(z + 4)^2 = -9$$

$$z + 4 = \pm 3i$$

$$z = -4 \pm 3i$$

So the roots of $z^3 + 9z^2 + 33z + 25 = 0$ are -1 , $-4 + 3i$ and $-4 - 3i$

$$\text{c} \quad -1 + (-4 + 3i) + (-4 - 3i) = -9$$

6 a If $3 + i$ is a root, then $3 - i$ is also a root.

$$\begin{aligned}
6 \text{ b } g(z) &= (z-6)(z-(3+i))(z-(3-i)) \\
&= (z-6)(z^2 - (3-i)z - (3+i)z + (3+i)(3-i)) \\
&= (z-6)(z^2 - 3z + iz - 3z - iz + 9 - 3i + 3i - i^2) \\
&= (z-6)(z^2 - 6z + 10) \\
&= z^3 - 6z^2 + 10z - 6z^2 + 36z - 60 \\
&= z^3 - 12z^2 + 46z - 60
\end{aligned}$$

So $c = 46$ and $d = -60$.

$$\begin{aligned}
7 \text{ h}(z) &= 2z^3 + 3z^2 + 3z + 1 \\
&= (2z+1)(az^2 + bz + c) \\
&= 2az^3 + (a+2b)z^2 + (2c+b)z + c
\end{aligned}$$

Equate z^3 coefficient:

$$2a = 2, \text{ so } a = 1$$

Equate z^2 coefficients:

$$a + 2b = 3, \text{ so } b = 1$$

Equate constants:

$$c = 1$$

$$0 = h(z)$$

$$= (2z+1)(z^2 + z + 1)$$

Either $2z+1=0 \Rightarrow z = -\frac{1}{2}$ or

$$z^2 + z + 1 = 0$$

Solve $z^2 + z + 1 = 0$ by completing the square:

$$\left(z + \frac{1}{2}\right)^2 - \frac{1}{4} + 1 = 0$$

$$\left(z + \frac{1}{2}\right)^2 = -\frac{3}{4}$$

$$z + \frac{1}{2} = \pm \frac{\sqrt{3}}{2}i$$

$$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

So the roots of $2z^3 + 3z^2 + 3z + 1 = 0$ are

$$-\frac{1}{2}, \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \text{ and } \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$8 \text{ a } f(z) = z^3 - 6z^2 + 28z + k$$

$$0 = f(2)$$

$$= (2)^3 - 6(2)^2 + 28(2) + k$$

$$= 8 - 24 + 56 + k$$

$$k = -40$$

b $z = 2$ is a root $\Rightarrow z - 2$ is a factor

$$\begin{array}{r}
z^2 - 4z + 20 \\
z-2 \overline{) z^3 - 6z^2 + 28z - 40} \\
\underline{z^3 - 2z^2} \\
-4z^2 + 28z \\
\underline{-4z^2 + 8z} \\
20z - 40 \\
\underline{20z - 40} \\
0
\end{array}$$

$$0 = f(z)$$

$$= (z-2)(z^2 - 4z + 20)$$

Either $z = 2$ or $z^2 - 4z + 20 = 0$

Solve $z^2 - 4z + 20 = 0$ by completing the square:

$$z^2 - 4z + 20 = 0$$

$$(z-2)^2 - 4 + 20 = 0$$

$$(z-2)^2 = -16$$

$$z-2 = \pm 4i$$

$$z = 2 \pm 4i$$

So the other roots of

$$z^3 - 6z^2 + 28z + 40 = 0 \text{ are}$$

$$2 + 4i \text{ and } 2 - 4i.$$

$$9 \quad z^4 - 16 = 0$$

$$(z^2 - 4)(z^2 + 4) = 0$$

$$(z-2)(z+2)(z^2 + 4) = 0$$

$$\text{Either } z-2=0 \Rightarrow z=2$$

$$\text{or } z+2=0 \Rightarrow z=-2$$

$$\text{or } z^2 + 4 = 0 \Rightarrow z^2 = -4$$

$$\Rightarrow z = \pm 2i$$

So the roots of $z^4 - 16 = 0$ are

$$2, -2, 2i \text{ and } -2i$$

$$\begin{aligned}
 \mathbf{10\ a} \quad z^4 - 12z^3 + 31z^2 + 108z - 360 \\
 &= (z^2 - 9)(z^2 + bz + c) \\
 &= z^4 + bz^3 + (-9 + c)z^2 - 9bz - 9c
 \end{aligned}$$

Equate z^3 coefficients:
 $b = -12$

Equate constants:
 $-9c = -360$, so $c = 40$

So $f(z) = (z^2 - 9)(z^2 - 12z + 40)$

$$\begin{aligned}
 \mathbf{b} \quad 0 = f(z) \\
 &= (z^2 - 9)(z^2 - 12z + 40)
 \end{aligned}$$

Either $z^2 - 9 = 0 \Rightarrow z = \pm 3$

Or $z^2 - 12z + 40 = 0$

$$\begin{aligned}
 (z - 6)^2 - 36 + 40 &= 0 \\
 (z - 6)^2 &= -4 \\
 z - 6 &= \pm 2i \\
 z &= 6 \pm 2i
 \end{aligned}$$

So the solutions to $f(z) = 0$ are
 $3, -3, 6 + 2i$ and $6 - 2i$

$\mathbf{11}$ If $g(2 + 3i) = 0$ then also $g(2 - 3i) = 0$

So $(z - (2 + 3i))(z - (2 - 3i))$ is a factor
of $g(z)$

$$\begin{aligned}
 \text{Now } (z - (2 + 3i))(z - (2 - 3i)) \\
 &= z^2 - (2 - 3i)z - (2 + 3i)z \\
 &\quad + (2 + 3i)(2 - 3i) \\
 &= z^2 - 2z + 3iz - 2z - 3iz + 4 - 6i + 6i - 9i^2 \\
 &= z^2 - 4z + 13
 \end{aligned}$$

So $z^2 - 4z + 13$ is a factor of $g(z)$

Use long division to find another factor:

$$\begin{array}{r}
 z^2 + 6z + 10 \\
 z^2 - 4z + 13 \overline{) z^4 + 2z^3 - z^2 + 38z + 130} \\
 \underline{z^4 - 4z^3 + 13z^2} \\
 6z^3 - 14z^2 + 38z \\
 \underline{6z^3 - 24z^2 + 78z} \\
 10z^2 - 40z + 130 \\
 \underline{10z^2 - 40z + 130} \\
 0
 \end{array}$$

$\mathbf{11}$ So $g(z) = (z^2 - 4z + 13)(z^2 + 6z + 10)$

$g(z) = 0 \Rightarrow$

either $(z^2 - 4z + 13) = 0 \Rightarrow z = 2 \pm 3i$

(these were the two solutions stated at the beginning of the question)

or $z^2 + 6z + 10 = 0$

$(z + 3)^2 - 9 + 10 = 0$

$(z + 3)^2 = -1$

$z + 3 = \pm i$

$z = -3 \pm i$

So the roots of $g(z) = 0$ are

$2 + 3i, 2 - 3i, -3 + i$ and $-3 - i$

$\mathbf{12\ a}$ If $z = 2 - 3i$ is a root then

$z = 2 + 3i$ is also a root

So $(z - (2 - 3i))(z - (2 + 3i))$ is a factor of
 $f(z)$

Now $(z - (2 - 3i))(z - (2 + 3i))$

$= z^2 - (2 + 3i)z - (2 - 3i)z$

$+ (2 - 3i)(2 + 3i)$

$= z^2 - 2z - 3iz - 2z + 3iz + 4 + 6i - 6i - 9i^2$

$= z^2 - 4z + 13$

So $z^2 - 4z + 13$ is a factor of $f(z)$

$\therefore f(z) = z^4 - 10z^3 + 71z^2 + Qz + 442$

$= (z^2 - 4z + 13)(z^2 + bz + c)$

$= z^4 + (-4 + b)z^3 + (c - 4b + 13)z^2$
 $+ (13b - 4c)z + 13c$

Equate z^3 coefficients:

$-4 + b = -10$, so $b = -6$

Equate constants:

$13c = 442$, so $c = 34$

Hence $(z^2 - 6z + 34)$ is a factor of $f(z)$

$\mathbf{12\ b}$ To find Q , equate z coefficients:

$13b - 4c = Q$

$13(-6) - 4(34) = Q$

$Q = -214$

$$12 \text{ c } 0 = f(z)$$

$$= (z^2 - 4z + 13)(z^2 - 6z + 34)$$

$$\text{either } (z^2 - 4z + 13) = 0 \Rightarrow z = 2 \pm 3i$$

(these were the two solutions stated at the beginning of the question)

$$\text{or } z^2 - 6z + 34 = 0$$

$$(z-3)^2 - 9 + 34 = 0$$

$$(z-3)^2 = -25$$

$$z-3 = \pm 5i$$

$$z = 3 \pm 5i$$

So the roots of $f(z) = 0$

are $2 - 3i$, $2 + 3i$, $3 + 5i$ and $3 - 5i$.

Challenge

$$z^5 + bz^4 + cz^3 + dz^2 + ez + f = 0 \quad (*)$$

The other two roots of (*) are $-2i$ and $1 - i$

For any pair of complex conjugate roots α and β of (*), the quadratic expression

$$(z - \alpha)(z - \beta) = z^2 - (\alpha + \beta)z + \alpha\beta$$

is a factor of (*)

Using $\alpha = 2i$ and $\beta = -2i$,

$$\alpha + \beta = 2i - 2i = 0$$

$$\alpha\beta = (2i)(-2i) = -4i^2 = 4$$

So one quadratic factor of (*) is

$$z^2 + 4 = 0$$

Using $\alpha = 1 + i$ and $\beta = 1 - i$,

$$\alpha + \beta = (1 + i) + (1 - i) = 2$$

$$\alpha\beta = (1 + i)(1 - i)$$

$$= 1(1 - i) + i(1 - i)$$

$$= 1 - i + i - i^2 = 2$$

So a second quadratic factor of (*) is

$$z^2 - 2z + 2 = 0$$

Also, since -2 is the only real root of (*), then

$(z + 2)$ is a linear factor of (*)

Hence,

$$0 = z^5 + bz^4 + cz^3 + dz^2 + ez + f$$

$$= (z + 2)(z^2 + 4)(z^2 - 2z + 2)$$

$$= (z^3 + 2z^2 + 4z + 8)(z^2 - 2z + 2)$$

$$= z^3(z^2 - 2z + 2) + 2z^2(z^2 - 2z + 2)$$

$$+ 4z(z^2 - 2z + 2) + 8(z^2 - 2z + 2)$$

$$= z^5 - 2z^4 + 2z^3 + 2z^4 - 4z^3 + 4z^2$$

$$+ 4z^3 - 8z^2 + 8z + 8z^2 - 16z + 16$$

$$= z^5 + 2z^3 + 4z^2 - 8z + 16$$

$$b = 0, c = 2, d = 4, e = -8, f = 16$$